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Existence and Characterization of Optimal Quadrature Formulas for a Certain Class of Differentiable Functions

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Let the integers $(\nu_k)_1^n$, $1 \leqslant \nu_k \leqslant r$, be fixed. We show that there exists a quadrature formula with nodes $a < x_1^* < \cdots < x_n^* < b$ of multiplicities $\nu_1, ..., \nu_n$, respectively, which has a minimal error in the Sobolev space $W_\infty^r[a, b]$ among all quadratures with nodes $(x_k)_1^n$, $a \leqslant x_1 < \cdots < x_n \leqslant b$, of the same multiplicities $(\nu_k)_1^n$.

1. INTRODUCTION

Let F be a class of sufficiently smooth functions defined in the interval $[a, b]$. The various variants of the problem of "optimal" approximation of the integral $I(f) = \int_a^b f(t)\, dt$ on the basis of a preassigned number of values of the integrand and its derivatives could be covered by the following two general formulations.

Let $\nu_1, ..., \nu_n$ be given positive integers. Construct a quadrature formula of the form

$$I(f) \approx \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k - 1} a_{k\lambda} f^{(\lambda)}(x_k) \tag{1.1}$$

which has

(i) as high a degree of exactness as possible;

(ii) a minimal error in the class F.

Problem (i) is a classical one. In the case $\nu_1 = \cdots = \nu_n = 1$ it has been solved by Gauss. The case of an arbitrary system of multiplicities $(\nu_k)_1^n$ was studied by Tschakaloff in his remarkable work [1].

Sard [2] and Nikolskiĭ [3] opened up a wide field of investigations devoted to problem (ii). In this paper we consider the question of existence and characterization of the solution of problem (ii), treating as avialable parameters

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both the coefficients and the nodes with fixed multiplicities. This question, which is central to the theory of quadrature formulas, has been attacked by many authors. Their efforts succeeded mosthly in the case when F is a class of analytic functions (see [4–6]).

In recent years, the Sobolev spaces $W_p^r[a, b]$,

$$W_p^r[a, b] := \{f \in C^{r-1}[a, b]: f^{(r-1)} \text{ abs. cont.}, f^{(r)} \in L_p[a, b]\}$$

became a touchstone for almost every new method in the theory of approximation. In spite of the importance of the existence problem and the popularity of the Sobolev spaces, the existence of optimal quadrature formulae of fixed type is known (see [7]) only in the special cases $r = 1, 2$ $(1 \leqslant p \leqslant \infty)$, or $\nu_1 = \cdots = \nu_n = \nu$, $\nu = r - 1, r - 2$ $(1 \leqslant p \leqslant \infty, r = 1, 2, \ldots)$. Schoenberg [8] and Karlin [9] announced without proofs existence and uniqueness theorems for the class $W_2^r[a, b]$ in the case $\nu_1 = \cdots = \nu_n = 1, 2$. (The existence was shown by Powell [10].) The existence of optimal quadrature formulas with simple nodes (i.e., $\nu_1 = \cdots = \nu_n = 1$) was proved recently for the classes $\tilde{W}_p^r$ $(r = 1, 2, \ldots, 1 \leqslant p \leqslant \infty)$ of periodic functions (see [11, 12]).

We show here the existence of optimal quadrature formulas in the class $W_\infty^r[a, b]$ for any admissible choice of the multiplicities of the nodes. The main result of our paper was announced in [13].

2. Definitions and Preliminary Results

Let $[a, b]$ be an interval of the real line and let r be a positive integer. Everywhere in this paper we shall write

$$\mathbf{x} = \begin{pmatrix} x_1, \ldots, x_n \\ \nu_1, \ldots, \nu_n \end{pmatrix} \tag{2.1}$$

to denote that $\mathbf{x}$ is a system of nodes $(x_k)_1^n$ with corresponding multiplicities $(\nu_k)_1^n$ such that $N = \nu_1 + \cdots + \nu_n \geqslant r$ and

$$a \leqslant x_1 < \cdots < x_n \leqslant b,$$
$$1 \leqslant \nu_k \leqslant r, \qquad k = 1, \ldots, n.$$

Let us denote by $\Omega(\nu_1, \ldots, \nu_n)$ the set of all systems $\mathbf{x}$ of the form (2.1). Given $\mathbf{x} \in \Omega(\nu_1, \ldots, \nu_n)$, we shall study the methods of approximation of the integral $I(f)$ in the class $W_\infty^r[a, b]$ which use only the information $T(\mathbf{x}; f) := \{f^{(\lambda)}(x_k), k = 1, \ldots, n, \lambda = 0, \ldots, \nu_k - 1\}$. Evidently any such method S is defined by a transformation of the set $\{T(\mathbf{x}; f): f \in W_\infty^r[a, b]\}$ into $\mathbb{R}$. Denote by $S(f)$ the approximate value of $I(f)$ given by the method S. We set, for simplicity of notation,

$$W = \{f \in W_\infty{}^r[a, b]: \|f^{(r)}\|_\infty \leqslant 1\}$$

($\|f\|_p$ will denote the L_p-norm of f in $[a, b]$, $1 \leqslant p \leqslant \infty$). The quantity

$$R(S; \mathbf{x}) = \sup\{|I(f) - S(f)|: f \in W\}$$

is called the error of the method S in the class $W_\infty{}^r[a, b]$. Let us denote $R(\mathbf{x}) = \inf\{R(S; \mathbf{x}): S\}$, where inf is extended over all admissible methods of approximation of the integral $I(f)$ that use only the information $T(\mathbf{x}; f)$. The method S_0 for which $R(S_0\,; \mathbf{x}) = R(\mathbf{x})$ is said to be a best method of integration in the class $W_\infty{}^r[a, b]$ on the basis of the information $T(\mathbf{x}; f)$. It follows from a general result of Smolyak [14] (see also [15] for the proof) that for every system $\mathbf{x} \in \Omega(\nu_1, ..., \nu_n)$ there is a linear best method of integration; i.e., there exist coefficients $\mathbf{a} = \{a_{k\lambda}\}$ such that

$$R(\mathbf{x}) = \sup\{|I(f) - S(\mathbf{a}, \mathbf{x}; f)|: f \in W\},$$

where

$$S(\mathbf{b}, \mathbf{x}; f) = \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k - 1} b_{k\lambda} f^{(\lambda)}(x_k).$$

The coefficients $\mathbf{a} = \mathbf{a}(\mathbf{x})$ are said to be best for the nodes $\mathbf{x}$. Smolyak [14] has also proved that

$$R(\mathbf{x}) = \sup\{I(f): f \in W(\mathbf{x})\}, \tag{2.2}$$

where $W(\mathbf{x}) = \{f \in W: f^{(\lambda)}(x_k) = 0,\ k = 1, ..., n,\ \lambda = 0, ..., \nu - 1\}$. Now it is easy to see that $R(\mathbf{x}) < \text{const}$ for every $\mathbf{x} \in \Omega(\nu_1, ..., \nu_n)$. Indeed, let $t_1, ..., t_r$ be the first r points in the sequence of nodes

$$\mathbf{x} = (\underbrace{x_1, ..., x_1}_{\nu_1}, x_2, ..., x_{n-1}, \underbrace{x_n \cdots x_n}_{\nu_n}).$$

By Newton's interpolation formula

$$f(t) = (t - t_1) \cdots (t - t_r) f[t, t_1, ..., t_r] \tag{2.3}$$

for every $f \in W(\mathbf{x})$, where $g[\tau_0, ..., \tau_m]$ denotes the divided difference of g based on the points $\tau_0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_m$. It is well-known (see [16] or [17]) that

$$g[\tau_0, ..., \tau_m] = \int_a^b u(t; \tau_0, ..., \tau_m)\, g^{(m)}(t)\, dt \tag{2.4}$$

for every $g \in W_\infty^r[a, b]$, where $u(t) = u(t; \tau_0, \ldots, \tau_m)$ is the divided difference of the function $(\cdot - t)_+^{m-1}/(m-1)!$ at the points $\tau_0, \ldots, \tau_m$. Moreover,

$$\begin{aligned} &u(t) > 0 \text{ for } t \in (\tau_0, \tau_m), \\ &u(t) = 0 \text{ for } t \notin [\tau_0, \tau_m], \\ &\int_a^b u(t)\,dt = 1/m! \end{aligned} \tag{2.5}$$

Then, it follows from (2.3)–(2.5) that $\sup\{f(t): f \in W(\mathbf{x})\} \leqslant (b-a)^r/r!$ and consequently, in view of (2.2),

$$R(\mathbf{x}) \leqslant (b-a)^{r+1}/r! \tag{2.6}$$

for every $\mathbf{x} \in \Omega(\nu_1, \ldots, \nu_n)$.

The estimate (2.6) implies that the best method must be exact for all polynomials of degree less or equal to $r-1$. For this reason we shall consider here only methods of the form

$$I(f) \approx S(\mathbf{b}, \mathbf{x}; f) \tag{2.7}$$

with coefficients $\mathbf{b}$ satisfying the requirement

$$I(P) = S(\mathbf{b}, \mathbf{x}; P) \tag{2.8}$$

for all $P \in \pi_{r-1}$. Here as elsewhere in this paper, π_m denotes the class of polynomials of degree m or less.

Hereafter we shall often be concerned with monosplines of the form

$$M(\mathbf{b}, \mathbf{x}; t) = \frac{(b-t)^r}{r!} - \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k - 1} b_{k\lambda} \frac{(x_k - t)_+^{r-\lambda-1}}{(r-\lambda-1)!} \tag{2.9}$$

satisfying the boundary conditions

$$M^{(j)}(\mathbf{b}, \mathbf{x}; a) = M^{(j)}(\mathbf{b}, \mathbf{x}; b) = 0, \qquad j = 0, \ldots, r-1. \tag{2.10}$$

There is a simple one-to-one correspondence between quadrature formulas and monosplines (see [8]). We shall briefly recall it here. Assuming that the coefficients $\mathbf{b}$ satisfy the requirement (2.8) and making use of Taylor's interpolation formula

$$f(x) = \sum_{k=0}^{r-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1} f^{(r)}(t)\,dt$$

we obtain the identity

$$I(f) - S(\mathbf{b}, \mathbf{x}; f) = \int_a^b M(\mathbf{b}, \mathbf{x}; t) f^{(r)}(t)\, dt \tag{2.11}$$

for every $f \in W_\infty{}^r[a, b]$, where the monospline $M(\mathbf{b}, \mathbf{x}; t)$ satisfies (2.10). Conversely, an arbitrary monospline (2.9) together with (2.10) induces a quadrature formula (2.7) which is evidently exact for $f \in \pi_{r-1}$. Application of Schwarz' inequality to (2.11) gives

$$| I(f) - S(\mathbf{b}, \mathbf{x}; f)| \leqslant \int_a^b | M(\mathbf{b}, \mathbf{x}; t)|\, dt \tag{2.12}$$

for every $f \in W$, provided $\mathbf{b}$ satisfies (2.8). Moreover, the equality in (2.12) holds only for functions $f \in W$ such that

$$f^{(r)}(t) = \operatorname{sign} M(\mathbf{b}, \mathbf{x}; t), \qquad t \in [a, b]. \tag{2.13}$$

By virtue of the optimality of the linear methods and the exactness of the best method for the class π_{r-1}, we conclude from (2.12) that

$$\begin{aligned} R(\mathbf{x}) &= \min \{\| M(\mathbf{b}, \mathbf{x}; \cdot)\|_1 : \mathbf{b} \text{ satisfies } (2.8)\} \\ &= \| M(\mathbf{a}, \mathbf{x}; \cdot)\|_1 , \end{aligned} \tag{2.14}$$

where the coefficients $\mathbf{a}$ are best for the nodes $\mathbf{x}$. So, the problem of construction of best quadrature formula with fixed nodes $\mathbf{x}$ reduces to the problem of best L_1-approximation of zero by monosplines of the form (2.9) satisfying (2.10).

The extremal element $\mathbf{a}$ in (2.14) is not unique in general. But it is a well-known fact in the theory of approximation that the function sign $M(\mathbf{a}, \mathbf{x}; t)$ is one and the same for all extremal systems $\mathbf{a}(\mathbf{x})$. Denote it by $\psi(\mathbf{x}; t)$. Since the monosplines have finite number of zeros, equality (2.13) shows that

$$f^{(r)}(t) = \psi(\mathbf{x}; t), \qquad t \in [a, b], \tag{2.15}$$

for $f \in W$ iff $| I(f) - S(\mathbf{a}, \mathbf{x}; f)| = R(\mathbf{x})$.

Let the multiplicities $(\nu_k)_1^n$ be given satisfying the inequalities

$$1 \leqslant \nu_k \leqslant r, \qquad k = 1,..., n. \tag{2.16}$$

DEFINITION. *We call the nodes* $\mathbf{x} \in \Omega(\nu_1 ,..., \nu_n)$ *optimal of the type* $(\nu_1 ,..., \nu_n)$ *in the class* $W_\infty{}^r[a, b]$ *if*

$$R(x) = \inf\{R(\mathbf{y}): \mathbf{y} \in \Omega(\nu_1 ,..., \nu_n)\} =: E(\nu_1 ,..., \nu_n).$$

Also, the best quadrature formula for the nodes $\mathbf{x}$ *is said to be optimal of the type* $(\nu_1 ,..., \nu_n)$ *in* $W_\infty{}^r[a, b]$.

For each system of multiplicities $(\nu_k)_1^n$ satisfying (2.16) we shall demonstrate that a $\mathbf{x} \in \Omega(\nu_1, ..., \nu_n)$ can be found such that $R(\mathbf{x}) = E(\nu_1, ..., \nu_n)$. We shall use a known existence theorem for best spline approximation with free knots (see [18]). In order to simplify our approach we introduce some notation.

Let us set

$$\Omega_N = \{\mathbf{y} = (\tau_1, ..., \tau_N) \in \mathbb{R}^N : a \leqslant \tau_1 \leqslant \cdots \leqslant \tau_N \leqslant b\}.$$

With every point $\mathbf{y} \in \Omega_N$ we associate a system of nodes $B_r(\mathbf{y})$ in the following way: If $\mathbf{y}$ has m distinct coordinates $y_1 < \cdots < y_m$ with multiplicities $\rho_1, ..., \rho_m$, respectively, then

$$B_r(\mathbf{y}) = \begin{pmatrix} y_1, ..., y_m \\ \mu_1, ..., \mu_m \end{pmatrix},$$

where $\mu_k = \min(\rho_k, r)$, $k = 1, ..., n$.

We shall write $\| \mathbf{y} \|$ instead of $\max_{1 \leqslant k \leqslant N} | \tau_k |$ for each point $\mathbf{y} = (\tau_1, ..., \tau_N) \in \mathbb{R}^N$.

Denote by $\bar{\Omega}(\nu_1, ..., \nu_n)$ the "closure" of $\Omega(\nu_1, ..., \nu_n)$, i.e., the set of all systems $\mathbf{y}_0$ for which there exist $\mathbf{y} \in \Omega_N$ and a sequence $\{\mathbf{y}^{(i)}\}_{i=1}^{\infty}$ in $\Omega(\nu_1, ..., \nu_n)$ such that $\lim_{i\to\infty} \| \mathbf{y}^{(i)} - \mathbf{y} \| = 0$ and $\mathbf{y}_0 = B_r(\mathbf{y})$. We shall show that

$$\inf\{R(\mathbf{y}) : \mathbf{y} \in \Omega(\nu_1, ..., \nu_n)\} = \inf\{R(\mathbf{y}) : \mathbf{y} \in \bar{\Omega}(\nu_1, ..., \nu_n)\}. \tag{2.17}$$

First we prove an auxiliary result.

LEMMA 1. *Let $\{\varphi_k^{(m)}\}_{k=1}^n$ be a linarly independent set in linear normed space H for $m = 0, 1, ...$ Let the real matrices*

$$\mathbf{B}_m = \begin{pmatrix} b_{11}^{(m)}, ..., b_{1n}^{(m)} \\ \cdots\cdots \\ b_{r1}^{(m)}, ..., b_{rn}^{(m)} \end{pmatrix}$$

be given such that the determinant $\Delta_r^{(m)}$ of the first r columns of $\mathbf{B}_m$ is not zero for $m = 0, 1, ...$ Suppose that

$$\lim_{m\to\infty} | b_{lj}^{(m)} - b_{lj}^{(0)} | = 0, \qquad l = 1, ..., r, j = 1, ..., n,$$

$$\lim_{m\to\infty} \| \varphi_k^{(m)} - \varphi_k^{(0)} \|_H = 0.$$

Denote by A_m the set of all real vectors $\mathbf{a} = (a_1, ..., a_n)$ such that $\mathbf{B}_m \mathbf{a} = 0$. Let $f \in H$ and

$$0 < E_n^{(m)} := \inf \left\{ \left\| f - \sum_{k=1}^n a_k \varphi_k^{(m)} \right\|_H : \mathbf{a} \in A_m \right\} < C$$

for $m = 0, 1, ...$ Then $E_n^{(0)} = \lim_{m\to 0} E_n^{(m)}$.

Proof. Since $\Delta_r^{(m)} \neq 0$, there exist linear functions $l_k^{(m)}(t_1, ..., t_{n-r})$, $k = 1,..., r$, such that $a_k = l_k^{(m)}(a_{r+1}, ..., a_n)$, $k = 1,..., r$, $m = 0, 1,...,$ for all $\mathbf{a} \in A_m$ and $\{l_k^{(m)}(t_1, ..., t_{n-r})\}$ converges uniformly to $l_k^{(0)}(t_1, ..., t_{n-r})$ on the unit ball of $\mathbb{R}^{n-r}$. Let $E_n^{(m)} = \| f - \sum_{k=1}^n a_k^{(m)} \varphi_k^{(m)} \|_H$, $m = 0, 1,....$ Note that

$$E_n^{(m)} = \inf \left\{ \left\| f - \sum_{k=1}^{r} l_k^{(m)}(c_{r+1}, ..., c_n)\, \varphi_k^{(m)} - \sum_{k=r+1}^{n} c_k \varphi_k^{(m)} \right\|_H : (c_{r+1}, ..., c_n) \in \mathbb{R}^{n-r} \right\}. \tag{2.18}$$

Our first claim is that the sequences $\{a_k^{(m)}\}_{m=1}^{\infty}$, $k = 1,..., n$, are bounded. Indeed, it follows from the equivalence of the norms in span$\{\varphi_1^{(m)},..., \varphi_n^{(m)}\}$ that there exists a constant $C_1 > 0$ such that

$$a_m := \max_{1 \leqslant k \leqslant n} | a_k^{(m)} | \leqslant C_1 \theta_m ,$$

where $\theta_m = \| f - \sum_{k=1}^n a_k^{(m)} \varphi_k^{(0)} \|_H$. Evidently, there is an index m_0 such that $m \geqslant m_0$ implies

$$\max_{1 \leqslant k \leqslant n} \| \varphi_k^{(m)} - \varphi_k^{(0)} \|_H \leqslant \frac{1}{2nC_1}.$$

Then, we have

$$a_m \leqslant C_1 \theta_m \leqslant C_1 \left\| f - \sum_{k=1}^{n} a_k^{(m)} \varphi_k^{(m)} \right\|_H + a_m/2$$

for $m \geqslant m_0$. Consequently $a_m \leqslant 2C_1 C$. Hence, after going to a subsequence if necessary, we may assume that

$$\lim_{m \to \infty} a_k^{(m)} = \alpha_k , k = 1,..., n$$

Next, according to (2.18),

$$E_n^{(m)} = \left\| f - \sum_{k=1}^{r} l_k^{(m)}(a_{r+1}^{(m)}, ..., a_n^{(m)})\, \varphi_k^{(m)} - \sum_{k=r+1}^{n} a_k^{(m)} \varphi_k^{(m)} \right\|_H$$

$$\leqslant \left\| f - \sum_{k=1}^{r} l_k^{(m)}(a_{r+1}^{(0)}, ..., a_n^{(0)})\, \varphi_k^{(m)} - \sum_{k=r+1}^{n} a_k^{(0)} \varphi_k^{(m)} \right\|_H .$$

Therefore

$$\lim_{m \to \infty} E_n^{(m)} = \left\| f - \sum_{k=1}^{r} l_k^{(0)}(\alpha_{r+1}, ..., \alpha_n)\, \varphi_n^{(0)} - \sum_{k=r+1}^{n} \alpha_k \varphi_k^{(0)} \right\|_H \leqslant E_n^{(0)}$$

and our assertion follows.

We recall that $\mathbf{a}(\mathbf{y})$ denotes a system of best coefficients for the nodes $\mathbf{y}$ in the sense of (2.14). With every $\mathbf{x} \in \Omega(\nu_1, ..., \nu_n)$ we associate the set $\Omega(\mathbf{x}) = \{\mathbf{y} \in \Omega_N : \| \mathbf{x} - \mathbf{y} \| \leqslant \Delta \mathbf{x}\}$, where $\Delta \mathbf{x} = \frac{1}{3} \min_{0 \leqslant k \leqslant n} | x_{k+1} - x_k |$, $x_0 = a$, $x_{n+1} = b$.

LEMMA 2. *Let the multiplicities $(\nu_k)_1^n$ satisfy (2.6) and let $\{\mathbf{y}^{(i)}\}$ be a sequence in $\Omega(\nu_1, ..., \nu_n)$ such that*

$$\lim_{i \to \infty} \| \mathbf{y}^{(i)} - \mathbf{x} \| = 0$$

for some $\mathbf{x} \in \Omega_N$. Suppose that $\mathbf{x}$ has m distinct components $a < x_1 < \cdots < x_m < b$ with multiplicities $\rho_1, ..., \rho_m$, respectively. Let $\mu_k = \min(\rho_k, r)$, $k = 1, ..., m$. Then the sequence $\{M(\mathbf{a}(\mathbf{y}^{(i)}), \mathbf{y}^{(i)}; t)\}_1^\infty$ converges uniformly to a monspline $M(t)$ of the form

$$M(t) = \frac{(b - t)^r}{r!} - \sum_{k=1}^{n} \sum_{\lambda=0}^{\mu_k - 1} c_{k\lambda} \frac{(x_k - t)_+^{r-\lambda-1}}{(r - \lambda - 1)!}$$

on each compact subset of $[a, b] \backslash \{x_1, ..., x_m\}$. Moreover, the coefficients $\mathbf{c} = \{c_{k\lambda}\}$ are best for the nodes $B_r(\mathbf{x})$.

Proof. The first (essential) part of our statement is proved in [18]. In order to derive it from [18], one needs only observe that the sequence $\{\| M(\mathbf{a}(\mathbf{y}^{(i)}), \mathbf{y}^{(i)}; \cdot) \|_1\}$ is uniformly bounded (see (2.6)). It remains to prove that the coefficients $\mathbf{c}$ of $M(t)$ are best for the nodes $B_r(\mathbf{x})$. To show this we shall apply Lemma 1.

Without loss of generality we may assume that $\| \mathbf{y}^{(i)} - \mathbf{x} \| \leqslant \Delta \mathbf{x}$ for $i = 1, 2, ...$ Denote by $y_{k1}^{(i)}, ..., y_{k,\rho_k}^{(i)}$ the coordinates τ of the point $\mathbf{y}^{(i)} \in \Omega_N$ for which $| x_k - \tau | \leqslant \Delta \mathbf{x}$. For simplicity let us set

$$\gamma(x; t) = (x - t)_+^{r-1}/(r - 1)!,$$

$$\gamma_{k\lambda}(t) = (\partial^\lambda / \partial x^\lambda) \gamma(x_k; t),$$

$$\gamma_{k\lambda}^{(i)}(t) = \gamma[y_{k1}^{(i)}, ..., y_{k,\lambda+1}^{(i)}; t].$$

It is easily verified (see [19]) that

$$\lim_{i \to \infty} \| \lambda! \gamma_{k\lambda}^{(i)} - \gamma_{k\lambda} \|_1 = 0$$

for $k = 1, ..., m$, $\lambda = 0, ..., \mu_k - 1$. We rewrite $M_i(t) = M(\mathbf{a}(\mathbf{y}^{(i)}), y^{(i)}; t)$ in the form

$$M_i(t) = \frac{(b - t)^r}{r!} - \sum_{k=1}^{m} \sum_{\lambda=0}^{\rho_k - 1} \lambda! \alpha_{k\lambda}^{(i)} \gamma_{k\lambda}^{(i)}(t).$$

Let the monospline

$$\tilde{M}_i(t) = \frac{(b-t)^r}{r!} - \sum_{k=1}^{n} \sum_{\lambda=0}^{\mu_k-1} A_{k\lambda}^{(i)} \gamma_{k\lambda}^{(i)}(t)$$

have a smallest L_1-norm among all monsplines of the same form with variable coefficients satisfying the boundary conditions (2.10). Evidently

$$\| M_i \|_1 \leqslant \| \tilde{M}_i \|_1, \qquad i = 1, 2, \dots. \tag{2.19}$$

On the other hand, applying Lemma 1 for $H = L_1[a, b]$, $\{\varphi_j^{(i)}\} = \{\lambda! \gamma_{k\lambda}^{(i)}\}$, $i = 1, 2, \dots$, $\{\varphi_j^{(0)}\} = \{\gamma_{k\lambda}\}$, and

$$A_i = \left\{ \mathbf{a} \in \mathbb{R}^N : I(g_j) = \sum_{k=1}^{m} \sum_{\lambda=0}^{\mu_k-1} a_{k\lambda} g_j[y_{k1}^{(i)}, \dots, y_{k,\lambda+1}^{(i)}], \text{ for } g_j(t) = t^j, \; j = 0, \dots, r-1 \right\}$$

we get

$$\lim_{i\to\infty} \| \tilde{M}_i \|_1 = \| M(\mathbf{a}(\mathbf{x}), \mathbf{x}; \cdot) \|_1 .$$

Then (2.19) implies $\| M \|_1 = \| M(\mathbf{a}(\mathbf{x}), \mathbf{x}; \cdot)\|_1$. The proof is completed.

As an immediate consequence of the above lemma and the relation $R(\mathbf{x}) = \| M(\mathbf{a}(\mathbf{x}), \mathbf{x}; \cdot)\|_1$ we get (2.17).

3. Properties of the Extremal Function

First we recall that a perfect spline of degree r with knots $\xi_1 < \cdots < \xi_k$ is a function of the form

$$\sum_{i=0}^{r-1} a_i t^i + c\left(t^r + 2 \sum_{i=1}^{k} (-1)^i (\xi_i - t)_+^r\right),$$

where $a_0, \dots, a_{r-1}$ and c are real constants. The following lemma (see [22]) is a simple consequence of the main result in [20] (see also [21]).

LEMMA 3. *For every system of nodes $\mathbf{x} \in \Omega(\nu_1, \dots, \nu_n)$ there exists a unique (up to multiplication by -1) perfect spline $\varphi(\mathbf{x}; t)$ of degree r having at most $N - r$ knots in (a, b) and satisfying the relations $\varphi(\mathbf{x}; t) \in W(\mathbf{x})$, $\| \varphi^{(r)}(\mathbf{x}; \cdot)\|_\infty = 1$.*

We need the following simple facts noted first in [22].

LEMMA 4. *Let $\mathbf{x} \in \Omega(\nu_1, \dots, \nu_n)$ and let $(\xi_i)_1^k$ be the knots of $\varphi(\mathbf{x}; t)$. If $\nu_m = r$ for some $1 \leqslant m \leqslant n$ then $x_m \notin (\xi_1, \dots, \xi_k)$.*

Remark 1. Repeated application of Rolle's theorem shows that $\varphi(\mathbf{x}; t)$ has precisely $N - r$ knots and precisely N zeros, counting multiplicities. Therefore $\varphi(\mathbf{x}; t) \neq 0$ for $t \neq x_i$ and $\varphi^{(\nu_i)}(\mathbf{x}; x_i) \neq 0$, $i = 1,..., n$.

LEMMA 5. *Let* $\mathbf{x} \in \Omega(\nu_1, ..., \nu_n)$. *Then*

$$|f(t)| \leqslant |\varphi(\mathbf{x}; t)|, \qquad t \in [a, b],$$

for every function $f \in W(\mathbf{x})$.

Further we shall often assume that the multiplicities $(\nu_k)_1^n$ satisfy the requirement

$$\begin{aligned} &1 \leqslant \nu_k \leqslant r, \qquad k = 1,..., n; \\ &\text{if } \nu_k < r \qquad \text{then } \nu_k \text{ is an even number.} \end{aligned} \tag{3.1}$$

Equation (3.1) will be referred to as evenness condition.

Now using the known result stated above we shall prove the following

LEMMA 6. *Let the multiplicities* $(\nu_k)_1^n$ *satisfy* (3.1). *Then, for every system of nodes* $\mathbf{x} \in \Omega(\nu_1, ..., \nu_n)$ *there exists exactly one function* $F(\mathbf{x}; t) \in W(\mathbf{x})$ *such that* $R(\mathbf{x}) = \int_a^b F(\mathbf{x}; t)\, dt$. *Furthermore*

(a) $|F^{(r)}(\mathbf{x}; t)| = 1$ *for each* $t \in [a, b]$;

(b) $F^{(r)}(\mathbf{x}; t)$ *has precisely* $N - r$ *sign changes when* r *is an even number and* $N - r + j$ *changes when* r *is odd, where* j *is the number of the multiplicities in the sequence* $\nu_1, ..., \nu_n$ *which are equal to* r;

(c) $F(\mathbf{x}; t) > 0$ *for all* $t \notin (x_1, ..., x_n)$;

(d) *The function* $F^{(\nu_k)}(\mathbf{x}; t)$ *is discontinuous at* x_k *if* r *is an odd number and* $\nu_k = r$;

(e) $F^{(\nu_k)}(\mathbf{x}; x_k) > 0$ *when* ν_k *is even.*

Proof. We shall show that $F(\mathbf{x}; t) = |\varphi(\mathbf{x}; t)|$. First we observe that the function $F(\mathbf{x}; t)$ defined as above belongs to the class $W_\infty^r[a, b]$. Indeed, $F(\mathbf{x}; t)$ has the same differential properties as the function $\varphi(\mathbf{x}; t)$ at every $t \in [a, b]$ excepting the points at which $\varphi(\mathbf{x}; t)$ changes its sign. Evidently, this occurs only for $t = x_k$ when ν_k is an odd number. According to assumption (3.1), ν_k is odd iff $\nu_k = r$ and r is odd. In this case, x_k is a point of discontinuity of $F^{(r)}(\mathbf{x}; t)$. By virtue of Lemma 4, x_k does not coincide with any knot of $\varphi(\mathbf{x}; t)$, i.e., x_k is an additional, newly introduced knot of the spline $F(\mathbf{x}; t)$. This proves properties (b) and (d). We see also that

$$F^{(\lambda)}(\mathbf{x}; x_k + 0) = F^{(\lambda)}(\mathbf{x}; x_k - 0) = 0 \qquad \text{for} \quad \lambda = 0,..., r - 1$$

if $\nu_k = r$. This implies that $F(\mathbf{x}; t)$ is $r - 1$ times continuously differentiable at the point x_k. Hence, $F(\mathbf{x}; \cdot) \in W_\infty^r[a, b]$. Adding the equalities

$$F^{(\lambda)}(\mathbf{x}; x_k) = 0, \qquad k = 1,..., n, \quad \lambda = 0,..., \nu_k - 1,$$

we conclude that $F(\mathbf{x}; \cdot) \in W(\mathbf{x})$.

Assertions (a) and (c) follow at once from the definition of $F(\mathbf{x}; t)$.

Next, making use of (2.2) and Lemma 5, we obtain

$$R(\mathbf{x}) = \sup \{I(f) : f \in W(\mathbf{x})\} \leqslant \int_a^b |\varphi(\mathbf{x}; t)| \, dt.$$

Since $F(\mathbf{x}; t) = |\varphi(\mathbf{x}; t)| \in W(\mathbf{x})$, we get $R(\mathbf{x}) = \int_a^b F(\mathbf{x}; t) \, dt$. Let us assume that there exists another function $F_1 \in W(\mathbf{x})$ for which $R(\mathbf{x}) = I(F_1)$. Thus, there is a point $t_0 \in [a, b]$ such that $|F_1(t_0)| > |\varphi(\mathbf{x}; t_0)|$. This inequality contradicts Lemma 5. The uniqueness of the extremal function is proved.

It remains to show that $F(\mathbf{x}; \cdot)$ satisfies (e). Clearly, $F^{(\nu_k)}(\mathbf{x}; x_k) \neq 0$ for even ν_k, by Remark 1. Now suppose that $F^{(\nu_k)}(\mathbf{x}; x_k) < 0$ for some even ν_k. Therefore, there is a positive number ϵ such that $F^{(\nu_k)}(\mathbf{x}; t) < 0$ for $t \in [x_k - \epsilon, x_k + \epsilon]$. Then, using Taylor's formula, we get

$$F(\mathbf{x}; t) = \frac{1}{(\nu_k - 1)!} \int_{x_k}^t (t - \tau)_+^{\nu_k - 1} F^{(\nu_k)}(\mathbf{x}; \tau) \, d\tau < 0$$

for each $t \in [x_k, x_k \in \epsilon]$, which contradicts (c). This completes the proof.

Next we show a continuous dependence of $F(\mathbf{x}; t)$ on the nodes $\mathbf{x}$.

LEMMA 7. *Let $\{\mathbf{y}^{(m)}\}$ be a sequence in Ω_N such that the multiplicities of the nodes in $\mathbf{y}^{(m)}$ $(m = 1, 2,...)$ satisfy the evenness condition (see (3.1)). Suppose that $\lim_{m\to\infty} \| \mathbf{y}^{(m)} - \mathbf{x} \| = 0$ for some $\mathbf{x} \in \Omega_N$. Then*

$$\lim_{m\to\infty} \| \psi(\mathbf{y}^{(m)}; \cdot) - \psi(B_r(\mathbf{x}); \cdot) \|_1 = 0,$$

$$\lim_{m\to\infty} \| F^{(j)}(\mathbf{y}^{(m)}; \cdot) - F^{(j)}(B_r(\mathbf{x}); \cdot) \|_{C[a,b]} = 0$$

for $j = 0,..., r - 1$.

Proof. The first assertion is an immediate consequence of Lemma 2. Now we shall prove the second one. By Taylor's interpolation formula

$$|P^{(\lambda)}(x_k) - P_m^{(\lambda)}(x_k)| \leqslant |F^{(\lambda)}(\mathbf{x}; x_k) - F^{(\lambda)}(\mathbf{y}^{(m)}; x_k)| + \frac{(b - a)^{r-\lambda-1}}{(r - \lambda - 1)!} \int_a^b |F^{(r)}(\mathbf{x}; t) - F^{(r)}(\mathbf{y}^{(m)}; t)| \, dt,$$

where

$$P(t) = \sum_{k=0}^{r-1} F^{(\lambda)}(\mathbf{x}; a)(t-a)^k/k!,$$

$$P_m(t) = \sum_{k=0}^{r-1} F^{(\lambda)}(\mathbf{y}^{(m)}; a)(t-a)^k/k!.$$

Since $F(\mathbf{y}^{(m)}; t) \in W(\mathbf{y}^{(m)})$ and $F(\mathbf{x}; t) \in W(\mathbf{x})$ we have

$$| F^{(\lambda)}(\mathbf{y}^{(m)}; x_k) - F^{(\lambda)}(\mathbf{x}; x_k)| \leqslant C \| \mathbf{x} - \mathbf{y}^{(m)} \|$$

for $\lambda = 0, 1,..., \min(\nu_k, r) - 1$, where C is a positive constant. In addition, as we note,

$$\lim_{m\to\infty} \| F^{(r)}(\mathbf{x}; \cdot) - F^{(r)}(\mathbf{y}^{(m)}; \cdot)\|_1 = 0.$$

Therefore, there exists $m_0 > 0$ such that

$$| P^{(\lambda)}(x_k) - P_m^{(\lambda)}(x_k)| \leqslant 2C \| \mathbf{x} - \mathbf{y}^{(m)} \| \tag{3.2}$$

for all $m \geqslant m_0$ and $k = 1,..., n$, $\lambda = 0,..., \min(\nu_k, r) - 1$. Since $N = \nu_1 + \cdots + \nu_n \geqslant r$, inequality (3.2) implies

$$\lim_{m\to\infty} \| P - P_m \|_{C[a,b]} = 0$$

and consequently $\lim_{m\to\infty} \| P^{(j)} - P_m^{(j)} \|_{C[a,b]} = 0$ for $j = 0,..., r-1$. Then our assertion follows at once from Taylor's formula.

4. Partial Characterization of the Optimal Quadrature Formulas

THEOREM 1. *Let the multiplicities* $(\nu_k)_1^n$ *satisfy the evenness condition* (3.1). *Suppose that the nodes* (2.1) *are optimal of the type* $(\nu_1,..., \nu_n)$ *in* $W_\infty^r[a, b]$. *Then* $a < x_1$ *and* $x_n < b$.

Proof. Let us assume that $a = x_1$ and let ϵ be an arbitrary positive number. Observe that the change

$$\tau = \tau(t) = b - \frac{b-t}{b-a}(b - a + \epsilon)$$

transforms the interval $[a, b]$ into $[a - \epsilon, b]$. Then the nodes

$$\mathbf{z} = \begin{pmatrix} z_1, ..., z_n \\ \nu_1, ..., \nu_n \end{pmatrix},$$

where $z_k = \tau(x_k)$, $k = 1,..., n$, will be optimal of the type $(\nu_1 ,..., \nu_n)$ in $W_\infty{}^r[a - \epsilon, b]$. Moreover, the function

$$\left(\frac{b - a + \epsilon}{b - a}\right)^r F\left(\mathbf{x}; b - \frac{b - a}{b - a + \epsilon} (b - \tau)\right)$$

must coincide with the spline $F(\mathbf{z}; \tau)$. Let R_ϵ denote the error of the optimal quadrature formula of the type $(\nu_1 ,..., \nu_n)$ in $W_\infty{}^r[a - \epsilon, b]$. Then, from Lemma 6,

$$R_\epsilon = \int_a^b F(\mathbf{z}; \tau)\, d\tau = (1 + \epsilon/(b - a))^{r+1} R(\mathbf{x}). \tag{4.1}$$

On the other hand, the best quadrature formula with nodes $\mathbf{x}$ for the class $W_\infty{}^r[a - \epsilon, b]$ has an error $R_\epsilon(\mathbf{x})$ which, according to Lemma 6, is defined by the formula

$$\begin{aligned} R_\epsilon(\mathbf{x}) &= \int_{a-\epsilon}^b F(\mathbf{x}; t)\, dt = R(\mathbf{x}) + \int_{a-\epsilon}^a F(\mathbf{x}; t)\, dt \\ &= R(\mathbf{x}) + \int_{a-\epsilon}^a \left[F(\mathbf{x}; a) + F'(\mathbf{x}; a)(t - a) + \int_a^t (t - \tau)\, F''(\mathbf{x}; \tau)\, d\tau\right] dt. \end{aligned}$$

Now, using the assumption $a = x_1$ and the equality $F'(\mathbf{x}; x_1) = \cdots = F^{(\nu_1-1)}(\mathbf{x}; x_1) = 0$, $\nu_1 \geqslant 2$, we get

$$R_\epsilon(\mathbf{x}) = R(\mathbf{x}) - \tfrac{1}{2} \int_{a-\epsilon}^a (\tau - a + \epsilon)^2 F''(\mathbf{x}; \tau)\, d\tau = R(\mathbf{x}) + O(\epsilon^3).$$

This and (4.1) give for a sufficiently small ϵ that $R_\epsilon(\mathbf{x}) < R_\epsilon$, which contradicts the definition of R_ϵ. Therefore $a < x_1$. In a similar way one could show that $x_n < b$. The theorem is proved.

THEOREM 2. *Let the multiplicities $(\nu_k)_1^n$ satisfy the evenness condition* (3.1). *Suppose that the nodes* $\mathbf{x}$ *are optimal of the type $(\nu_1 ,..., \nu_n)$ in the class $W_\infty{}^r[a, b]$. Let* $\mathbf{a} = \{a_{k\lambda}\}$ *be the best coefficients for the nodes* $\mathbf{x}$. *Then*

$$a_{k,\nu_k-1} = 0, \qquad a_{k,\nu_k-2} > 0 \qquad \text{if } \nu_k \text{ is even},$$
$$a_{k,\nu_k-1} > 0 \qquad \text{if } \nu_k \text{ is odd}.$$

Proof. Suppose that ν_k is even. According to Theorem 1, $a < x_k < b$. With every real h, $|h| \leqslant \Delta\mathbf{x}$, we associate the nodes

$$\mathbf{x}_h = \begin{pmatrix} x_1 ,..., x_{k-1} , x_k + h, x_{k+1} ,..., x_n \\ \nu_1 ,..., \nu_{k-1} , \nu_k , \nu_{k+1} ,..., \nu_n \end{pmatrix}.$$

Since the $\mathbf{x}$ are optimal,

$$R(\mathbf{x}) \leqslant R(\mathbf{x}_h). \tag{4.2}$$

On the other hand $F(\mathbf{x}_h\,; t) \in W$. Therefore

$$R(\mathbf{x}_h) - \sum_{k=0}^{\nu_k-1} a_{k\lambda} F^{(\lambda)}(\mathbf{x}!\,; x_k) \leqslant R(\mathbf{x}). \tag{4.3}$$

In view of assertion (e) of Lemma 6, there exists a number $\epsilon > 0$ such that $F^{(\nu_k)}(\mathbf{x}; t) > 0$ for all $t \in [x_k - \epsilon, x_k + \epsilon]$. Then, on the basis of Lemma 7, there exist h_0, $0 < h_0 < \Delta\mathbf{x}$, and constants $C_1 > 0$, $C_2 > 0$ such that $C_1 < F^{(\nu_k)}(\mathbf{x}_h\,; t) < C_2$ for all $t \in [x_k - \epsilon, x_k + \epsilon]$ and $|h| \leqslant h_0$. The above inequalities and the condition $F^{(\lambda)}(\mathbf{x}_h\,; x_k + h) = 0$ ($\lambda = 0,..., r - 1$) imply

$$C_1 \,|\, h\,|^{\nu_k-\lambda}/(\nu_k - \lambda)! \leqslant |\, F^{(\lambda)}(\mathbf{x}_h\,; x_k)| \leqslant C_2 \,|\, h\,|^{\nu_k-\lambda}/(\nu_k - \lambda)!\,. \tag{4.4}$$

In addition, obviously,

$$\begin{aligned} \operatorname{sign} F^{(\nu_k-1)}(\mathbf{x}_h\,; x_k) &= -\operatorname{sign} h, \\ F^{(\nu_k-2)}(\mathbf{x}_h\,; x_k) &> 0. \end{aligned} \tag{4.5}$$

Now, let us assume for a moment that $a_{k,\nu_k-1} \neq 0$. We choose h to satisfy $\operatorname{sign} h = -\operatorname{sign} a_{k,\nu_k-1}$. Then, taking into account (4.4) and (4.5), we get from (4.3)

$$R(\mathbf{x}_h) - \sum_{k=0}^{\nu_k-2} |\, a_{k\lambda}\,|\, C_2 \,|\, h\,|^{\nu_k-\lambda}/(\nu_k - \lambda)! + |\, a_{k,\nu_k-1}\,|\, C_1 \,|\, h\,| \leqslant R(\mathbf{x}),$$

which contradicts (4.2) for a sufficiently small h. Therefore $a_{k,\nu_k-1} = 0$. Then (4.3) gives

$$R(\mathbf{x}_h) - \sum_{k=0}^{\nu_k-3} |\, a_{k\lambda}\,|\, C_2 \,|\, h\,|^{\nu_k-\lambda}/(\nu_k - \lambda)!$$

$$- a_{k,\nu_k-2} F^{(\nu_k-2)}(\mathbf{x}_h\,; x_k) \leqslant R(\mathbf{x}).$$

In view of (4.5), the assumption $a_{k,\nu_k-2} < 0$ leads to contradiction. Therefore $a_{k,\nu_k-2} \geqslant 0$. Let us assume that $a_{k,\nu_k-2} = 0$. Consider the nodes

$$\mathbf{z} = \begin{pmatrix} x_1 ,..., x_{k-1} , x_k , x_{k+1} ,..., x_n \\ \nu_1 ,..., \nu_{k-1} , \nu_k - 2, \nu_{k+1} ,..., \nu_n \end{pmatrix}.$$

Since $a_{k,\nu_k-1} = a_{k,\nu_k-2} = 0$, we have $R(\mathbf{z}) = R(\mathbf{x})$. Then $I(F(\mathbf{x}; \cdot)) = I(F(\mathbf{z}; \cdot))$. This implies $F(\mathbf{x}; t) = F(\mathbf{z}; t)$, according to the unicity of the extremal function $F(\mathbf{z}; t)$. On the other hand, by Lemma 6, $F^{(\nu_k-2)}(\mathbf{z}; x_k) \neq 0$, $F^{(\nu_k)}(\mathbf{x}; x_k) = 0$. The contradiction completes the proof.

It remains to show that $a_{k,\nu_k-1} > 0$ if ν_k is an odd number. In this case we consider the function $F(\mathbf{y}; t)$ for

$$\mathbf{y} = \begin{pmatrix} x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n \\ \nu_1, \ldots, \nu_{k-1}, \nu_k - 1, \nu_{k+1}, \ldots, \nu_n \end{pmatrix}.$$

Evidently $F(\mathbf{y}; t) \in W$. Then

$$R(\mathbf{y}) - a_{k,\nu_k-1} F^{(\nu_k-1)}(\mathbf{y}; x_k) \leqslant R(\mathbf{x}).$$

But $\nu_k - 1$ is an even number. Then, according to Lemma 7, $F^{(\nu_k-1)}(\mathbf{y}; x_k) > 0$. Now, assuming that $a_{k,\nu_k-1} \leqslant 0$, we get $R(\mathbf{y}) \leqslant R(\mathbf{x})$, which is impossible since

$$R(\mathbf{y}) = \int_a^b F(\mathbf{y}; t)\, dt < \int_a^b F(\mathbf{x}; t)\, dt = R(\mathbf{x}).$$

The proof is complete.

5. Existence

First, we shall prove an auxiliary lemma.

Lemma 8. *Let h be an arbitrary positive number and $f \in C^r[\tau - h, \tau + h]$. Suppose that f has exactly r zeros in $[\tau - h, \tau + h]$ and $f(\tau - h) = f(\tau + h) = 0$. If $0 < m < f^{(r)}(t) < M$ for all $t \in [\tau - h, \tau + h]$, then $\beta - \alpha > m^{1/2}(Mr!\, 2^{r-2})^{-1/2} \cdot h$, where α, β are the zeros of $f^{(r-2)}(t)$ in $[\tau - h, \tau + h]$.*

Proof. By Rolle's theorem $f^{(k)}(t)$ has exactly $r - k$ zeros in $[\tau - h, \tau + h]$. Denote them by $\{t_{kj}\}_{j=1}^{r-k}$ $(t_{k1} \leqslant \cdots \leqslant t_{k,r-k})$. Evidently $f^{(k)}(x) = \int_{t_{k2}}^{x} f^{(k+1)}(t)\, dt$ for $k \leqslant r - 2$ and consequently

$$\max_{t_{k1} \leqslant x \leqslant t_{k,r-k}} | f^{(k)}(x) | \leqslant (t_{k,r-k} - t_{k1}) \max_{t_{k+1,1} \leqslant x \leqslant t_{k+1,r-k-1}} | f^{(k+1)}(x) |. \tag{5.1}$$

Let ξ be the unique zero of $f^{(r-1)}(t)$ in $[\tau - h, \tau + h]$. Then

$$\max_{\alpha \leqslant x \leqslant \beta} | f^{(r-2)}(x) | = | f^{(r-2)}(\xi) | \leqslant (\beta - \alpha)^2 M/4.$$

If $t_{k1} \leqslant \xi \leqslant t_{k,r-k}$ with $k = 0, \ldots, r - 2$, then repeated use of (5.1) gives

$$| f^{(k)}(\xi)| \leqslant (2h)^{r-k-2}(\beta - \alpha)^2 M/4, \qquad k = 0, \ldots, r - 2. \tag{5.2}$$

Now suppose that $\xi \leqslant \tau$. By Taylor's formula

$$f(x) = \sum_{k=1}^{r-1} f^{(k)}(\xi)(x-\xi)^k/k! + \frac{1}{(r-1)!}\int_\xi^x (x-t)^{r-1} f^{(r)}(t)\,dt.$$

Next, using (5.2) and the assumptions of the lemma, we get for $x \geqslant \tau$,

$$\begin{aligned} f(x) &\geqslant \frac{m}{(r-1)!}\int_\xi^x (x-t)^{r-1}\,dt - \sum_{k=0}^{r-2} | f^{(k)}(\xi)| | x-\xi |^k/k! \\ &> m(x-\tau)^r/r! - M(\beta-\alpha)^2 (2h)^{r-2}. \end{aligned}$$

In the special case $x = \tau + h$, the above inequality gives $0 = f(\tau+h) > mh^r/r! - M(\beta-\alpha)^2(2h)^{r-2}$ and our assertion follows immediately.

Now suppose that $\tau \leqslant \xi$. Let $x \leqslant \tau$. In a similar fashion we obtain

$$f(x) < -m(\tau - x)^r/r! + M(\beta-\alpha)^2(2h)^{r-2}$$

for odd r, and

$$f(x) > m(x-\tau)^r/r! - M(\beta-\alpha)^2(2h)^{r-2}$$

for even r. This, together with the assumption $f(\tau - h) = 0$, yields $M(\beta-\alpha)^2(2h)^{r-2} > mh^r/r!$. The lemma is proved.

THEOREM 3. *Suppose that r and N are arbitrary positive integers such that $N \geqslant r$. Then for every system of multiplicities $(\nu_k)_1^n$ satisfying $\sum_{k=1}^n \nu_k = N$, $1 \leqslant \nu_k \leqslant r$, $k = 1,..., n$, there exists an optimal quadrature formula of the type $(\nu_1,..., \nu_n)$ in the class $W_\infty{}^r[a, b]$. Furthermore, the coefficients $\mathbf{a} = \{a_{k\lambda}\}$ of the optimal quadrature formula satisfy the relations*

$$\begin{aligned} &a_{k,\nu_k-1} = 0, \quad a_{k,i} > 0, \quad j = 0, 2,..., \nu_k - 2, \quad \textit{for even } \nu_k, \\ &a_{k,j} > 0, \qquad\qquad\qquad j = 0, 2,..., \nu_k - 1, \quad \textit{for odd } \nu_k. \end{aligned}$$

Proof. First we consider the case when the multiplicities $(\nu_k)_1^n$ satisfy the evenness condition (3.1). Let $\{\mathbf{y}^{(i)}\}_1^\infty$ be a minimizing sequence in $\Omega(\nu_1,..., \nu_n)$, i.e., such that $\lim_{i\to\infty} R(\mathbf{y}^{(i)}) = E(\nu_1,..., \nu_n)$. After going to a subsequence if necessary, we may assume that $\lim_{i\to\infty} \| \mathbf{y}^{(i)} - \mathbf{z} \| = 0$ for some $\mathbf{z} \in \Omega_N$. Suppose that $\mathbf{z}$ has m distinct components $x_1 < \cdots < x_m$ of multiplicities $\rho_1,..., \rho_m$, respectively. Let us set $\mu_k = \min(\rho_k, r)$, $k = 1,..., m$. Set $\mathbf{x} = B_r(\mathbf{z})$. By virtue of Lemma 7, $E(\nu_1,..., \nu_n) = \lim_{i\to\infty} R(\mathbf{y}^{(i)}) = R(\mathbf{x})$. According to (2.17), $R(\mathbf{x}) \leqslant R(\mathbf{y}), \mathbf{y} \in \bar{\Omega}(\nu_1,..., \nu_n)$. Since $\Omega(\mu_1,..., \mu_m) \subset \bar{\Omega}(\nu_1,..., \nu_n)$ we see that the nodes $\mathbf{x}$ are optimal of the type $(\mu_1,..., \mu_m)$. Then, by Theorem 1, $a < x_1 < \cdots < x_m < b$. Evidently $m \leqslant n$. Let us assume that $m < n$. Then, there is an index k, $1 \leqslant k \leqslant m - 1$. such that $\rho_k = \nu_k + \nu_{k+1} + \cdots + \nu_{k+j}$ for some $j \geqslant 1$. We observe that

$\max(\nu_k, ..., \nu_{k+j}) < r$. Indeed, suppose that one of the numbers $\nu_k, ..., \nu_{k+j}$, say ν_k, is equal to r. Then we construct the nodes

$$\mathbf{y} = \begin{pmatrix} x_1, ..., x_{k-1}, x_k, t_0, x_{k+1}, ..., x_m \\ \mu_1, ..., \mu_{k-1}, \nu_k, \nu_{k+1}, \mu_{k+1}, ..., \mu_m \end{pmatrix},$$

where $t_0 \notin (x_1, ..., x_m)$. Since $\mathbf{y} \in \bar{\Omega}(\nu_1, ..., \nu_n)$ we have

$$R(\mathbf{x}) \leqslant R(\mathbf{y}). \tag{5.3}$$

On the other hand, in view of Lemma 5, $|\varphi(\mathbf{y}; t)| \leqslant |\varphi(\mathbf{x}; t)|$. Moreover, the above inequality is strict in a neighborhood of the point t_0. Then $R(\mathbf{y}) = \int_a^b |\varphi(\mathbf{y}; t)| \, dt < \int_a^b |\varphi(\mathbf{x}; t)| \, dt = R(\mathbf{x})$, which contradicts (5.3). Therefore $\max(\nu_k, ..., \nu_{k+j}) < r$.

Let us set $q = 2[(\mu_k + 1)/2]$. Here $[\cdot]$ is the greatest integer function. For any $h \geqslant 0$ and $\tau \in [x_k - h, x_k + h]$, τ being a parameter to be chosen later, we denote by $\mathbf{x}(h)$ the nodes

$$\mathbf{x}(h) = \begin{pmatrix} x_1, ..., x_{k-1}, \tau - h, \tau + h, x_{k+1}, ..., x_m \\ \mu_1, ..., \mu_{k-1}, \nu_k, q - \nu_k, \mu_{k+1}, ..., \mu_m \end{pmatrix}.$$

We claim that for all sufficiently small values of h there exists a point $\tau = \tau(h) \in [x_k - h, x_k + h]$ such that the function $F^{(q-1)}(\mathbf{x}(h); t)$ changes its sign in x_k. To prove this we consider the cases $q \leqslant r$ and $q = r + 1$ separately. Suppose that $q \leqslant r$. Then q is an even number. According to Lemma 6, there exists $\epsilon > 0$ such that $F^{(q)}(\mathbf{x}; t) > 0$ for all $t \in [x_k - \epsilon, x_k + \epsilon]$. By virtue of Lemma 7, there exists $0 < h_0 < \min\{\Delta\mathbf{x}, \epsilon/2\}$ such that

$$F^{(q)}(\mathbf{x}(h); t) > 0 \tag{5.4}$$

for all $t \in [x_k - \epsilon, x_k + \epsilon]$ and $h \leqslant h_0$. Since $F(\mathbf{x}(h); t)$ has q zeros in $[\tau - h, \tau + h]$,Rolle's theorem and (5.4) imply that $F^{(\lambda)}(\mathbf{x}(h); t)$ has precisely $q - \lambda$ zeros in $[\tau - h, \tau + h]$ for every $h \leqslant h_0$ and $\lambda = 0, ..., q$. Now suppose that h is fixed and $h \leqslant h_0$. Denote by $\xi(\tau)$ the unique zero of the function $F^{(q-1)}(\mathbf{x}(h); t)$ in $[\tau - h, \tau + h]$. It is easily seen on the basis of Lemma 7 that $\xi(\tau)$ is a continuous function of τ in $[x_k - h, x_k + h]$. In addition, repeated application of Rolle's theorem shows that $\xi(x_k - h) < x_k$ and $\xi(x_k + h) > x_k$. Therefore there exists a point $\tau \in [x_k - h, x_k + h]$ such that $\xi(\tau) = x_k$. Our claim is proved in the case $q \leqslant r$. Now suppose that $q = r + 1$. Obviously this occurs when $\mu_k = r$ and r is an odd number. By Lemma 6 there exists a number $\epsilon > 0$ such that $F(\mathbf{x}; t)$ has no other knots in $[x_k - \epsilon, x_k + \epsilon]$ excepting x_k. Since $F(\mathbf{x}; t)$ and $F(\mathbf{x}(h); t)$ have one and the same number of knots, it follows easily from Lemma 7 that $F(\mathbf{x}(h); t)$

has precisely one knot in $[x_k - \epsilon, x_k + \epsilon]$ for all sufficiently small h (say for all $h \leqslant h_0$). Given a h, we denote by $\xi(\tau)$ the knot of $F(\mathbf{x}(h); t)$ in $[x_k - \epsilon, x_k + \epsilon]$. Lemma 7 implies that $\xi(\tau)$ is a continuous function of τ for fixed h. Then, as above, we show that there is a point $\tau \in [x_k - \epsilon, x_k + \epsilon]$ such that $\xi(\tau) = x_k$. In what follows we choose τ in this way. It goes without saying that $h \leqslant h_0$ is assumed. We saw that the nodes $\mathbf{x}$ are optimal of the type $(\mu_1, ..., \mu_m)$ in the class $W_\infty{}^r[a, b]$. Moreover, $R(\mathbf{x}) \leqslant R(\mathbf{y})$ for all $\mathbf{y} \in \bar{\Omega}(\nu_1, ..., \nu_n)$. Hence

$$R(\mathbf{x}) \leqslant R(\mathbf{x}(h)) \tag{5.5}$$

for all $h \leqslant h_0$. Denote by $\mathbf{a} = \{a_{k\lambda}\}$ the best coefficients for the nodes $\mathbf{x}$. Since $F(\mathbf{x}(h); t) \in W$, we have

$$\int_a^b F(\mathbf{x}(h); t)\, dt - \sum_{\lambda=0}^{\mu_k-1} a_{k\lambda} F^{(\lambda)}(\mathbf{x}(h); x_k) \leqslant R(\mathbf{x}). \tag{5.6}$$

The function $F^{(\lambda)}(\mathbf{x}(h); t)$ $(\lambda = 0, ..., \mu_k - 1)$ has $\mu_k - \lambda$ zeros at least in $[\tau - h, \tau + h]$. In addition, $F^{(\mu_k-1)}(\mathbf{x}(h); t)$ is absolutely continuous and $F^{(\mu_k)}(\mathbf{x}(h); t)$ is bounded over $[a, b]$. Therefore, using Newton's interpolation formula one can show that there exists a constant $C > 0$ such that

$$| F^{(\lambda)}(\mathbf{x}(h); x_k)| \leqslant C h^{\mu_k - \lambda} \tag{5.7}$$

for $\lambda = 0, ..., \mu_k - 1$.

Next we continue the proof of the theorem considering the cases of even μ_k and odd μ_k separately. Let μ_k be an even number. Then $q = \mu_k$. Denote by α, β the zeros of $F^{(q-2)}(\mathbf{x}(h); t)$ in $[\tau - h, \tau + h]$. By Newton's interpolation formula

$$F^{(q-2)}(\mathbf{x}(h); t) = (t - \alpha)(t - \beta) \int_\alpha^\beta u(s; \alpha, \beta)\, F^{(q)}(\mathbf{x}(h); s)\, ds.$$

Since $F^{(q-1)}(\mathbf{x}(h); x_k) = 0$, we get

$$\begin{aligned} | F^{(q-2)}(\mathbf{x}(h); x_k)| &= \max_{\alpha \leqslant t \leqslant \beta} | F^{(q-2)}(\mathbf{x}(h); t)| \\ &\geqslant \frac{(\beta - \alpha)^2}{4} \min_{\alpha \leqslant t \leqslant \beta} F^{(q)}(\mathbf{x}(h); t). \end{aligned}$$

Now we conclude from Lemma 7 and inequality (e) from Lemma 6 that there exists a constant $c_1 > 0$ such that $F^{(q)}(\mathbf{x}(h); t) > 4c_1$ in $[\alpha, \beta]$ for all sufficiently small h. Therefore

$$| F^{(q-2)}(\mathbf{x}(h); x_k)| \geqslant c_1(\beta - \alpha)^2. \tag{5.8}$$

According to the optimality of $\mathbf{x}$ we have

$$a_{k,\mu_k-1} = 0, \qquad a_{k,\mu_k-2} > 0. \tag{5.9}$$

Obviously

$$F^{(q-2)}(\mathbf{x}(h); t) < 0 \tag{5.10}$$

in (α, β), since q is even and $F(\mathbf{x}(h); t) \geqslant 0$ in $[a, b]$. Taking into account (5.7)–(5.10), we get from (5.6), $R(\mathbf{x}(h)) + a_{k,\mu_k-2}c_1h^2 + O(h^3) \leqslant R(\mathbf{x})$. This shows that $R(\mathbf{x}(h)) < R(\mathbf{x})$ for sufficiently small h, contradicting (5.5).

Now consider the case of odd μ_k. Then $\mu_k = r$, $a = r + 1$. It follows from (5.6) and (5.7) that

$$R(\mathbf{x}(h)) + a_{k,r-1}F^{(r-1)}(\mathbf{x}(h); x_k) + O(h^2) \leqslant R(\mathbf{x}). \tag{5.11}$$

The function $F(\mathbf{x}(h); t)$ has $r + 1$ zeros in $[\tau - h, \tau + h]$. By Rolle's theorem, $F^{(r-1)}(\mathbf{x}(h); t)$ has at least two zeros in $[\tau - h, \tau + h]$. It is easily seen on the basis of Lemmas 6 and 7 that $F^{(r-1)}(\mathbf{x}(h); t)$ actually has precisely two zeros in $[\tau - h, \tau + h]$ for sufficiently small h. Let us denote them by $t_1(h)$ and $t_2(h)$. The assumption that $F^{(r)}(\mathbf{x}(h); t)$ changes its sign at the point x_k gives $x_k = (t_1(h) + t_2(h))/2$ and $F^{(r-1)}(\mathbf{x}(h); x_k) = -(t_2(h) - t_1(h))/2$. Consider now the behavior of the distance $t_2(h) - t_1(h)$ when h tends to zero. First we observe that the spline function $g(t)$ of odd degree r is uniquely determined by the conditions $g(t) \geqslant 0$, $|g^{(r)}(t)| = 1$ for all t; $g(t)$ has $r + 1$ zeros at the points $\tau - h, \tau + h$ of multiplicities ν_k and $r + 1 - \nu_k$, respectively; $g(t)$ has exactly one knot in $(\tau - h, \tau + h)$. Denote by $g_0(t)$ the spline function satisfying the above conditions for $\tau = 0$ and $h = 1$. Therefore $F(\mathbf{x}(h); t) = h^r g_0((t - \tau)/h)$ for $t \in \tau - h, [\tau + h]$ in view of the uniqueness of the spline g. It follows from this relation that $t_2(h) - t_1(h) = \delta h$, where δ is the distance between the zeros of $g_0^{(r-1)}(t)$. Thus, we get from (5.11)

$$R(\mathbf{x}(h)) + a_{k,r-1}\, \delta h/2 \leqslant R(\mathbf{x}).$$

As far as $a_{k,r-1} > 0$ is concerned, we obtain $R(\mathbf{x}(h)) < R(\mathbf{x})$ for sufficiently small h. This contradicts (5.5). Hence $m = n$. This entails $\mu_i = \nu_i$ for $i = 1,..., n$. So, the existence part of our theorem is proved in the case of multiplicities satisfying (3.1). In the general case we consider the multiplicities $\mu_k = \min(r, 2[(\nu_k + 1)/2]$, $k = 1,..., n$. Then $(\mu_k)_1^n$ are even or equal to r and the optimal quadrature formula of the type $(\mu_1 ,..., \mu_n)$ exists. But, according to Theorem 2, $a_{k,\mu_k-1} = 0$ for $\mu_k > \nu_k$. Therefore the same quadrature is optimal of the type $(\nu_1 ,..., \nu_n)$ also.

To prove the last assertion of the theorem we need the following result due to Micchelli [23]:

Let M be a monospline of the form

$$M(t) = \frac{t^r}{r!} + \sum_{i=0}^{r-1} a_i t^i + \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k-1} c_{k\lambda}(t - x_k)_+^{r-\lambda-1}.$$

Then $M(t)$ has at most $r + \sum_{k=1}^{n} (\nu_k + \sigma_k)$ zeros in $(-\infty, \infty)$ counting multiplicities, where $\sigma_k = 1$ if ν_k is odd and zero otherwise. Moreover, if $M(t)$ has the maximal number of zeros in $(-\infty, \infty)$ then $c_{k\lambda} < 0$, $\lambda = 0, 2,..., \nu_k - 1$, if ν_k is odd.

For the precise definition of a zero of multiplicity α, where α is allowed to be as large as $r + 1$, the reader is referred to [23]. This definition is chosen so that M changes sign if α is odd, and does not change sign if α is even.

Now suppose that the quadrature formula (1.1) is optimal of the type $(\nu_1,..., \nu_n)$. According to Theorem 2 we may assume without loss of generality that $(\nu_k)_1^n$ satisfy (3.1). In view of Lemma 6, $F(\mathbf{x}; t)$ has precisely $Z = \sum_{k=1}^{n} (\nu_k + \sigma_k)$ zeros in $[a, b]$ counting multiplicities as in [23]. Then a repeated application of Rolle's theorem shows that $F^{(r)}(\mathbf{x}; t)$ has at least $Z - r$ zeros in (a, b). But $F^{(r)}(\mathbf{x}; t) = \operatorname{sign} M(\mathbf{a}, \mathbf{x}; t)$. Adding the conditions (2.10), we conclude that $M(\mathbf{a}, \mathbf{x}; t)$ has at least $Z + r$ zeros in $(-\infty, \infty)$. By virtue of Theorem 2, $M(\mathbf{a}, x; t)$ can be rewritten in the form

$$M(\mathbf{a}, \mathbf{x}; t) = \frac{t^r}{r!} + \sum_{i=1}^{r-1} b_i t^i - \sum_{k=1}^{n} \sum_{\lambda=0}^{\mu_k-1} a_{k\lambda} \frac{(t - x_k)_+^{r-\lambda-1}}{(r - \lambda - 1)!},$$

where $\mu_k = \nu_k - 1 + \sigma_k$, $k = 1,..., n$. Evidently $(\mu_k)_1^n$ are odd. Then $M(\mathbf{a}, \mathbf{x}; t)$ has the maximal number of zeros in $(-\infty, \infty)$ and according to Micchelli's result, $a_{k\lambda} > 0$ for $k = 1,..., n$ and $\lambda = 0, 2,..., \nu_k + \sigma_k - 2$. The theorem is proved.

A careful tracing of the proof of Theorem 3 shows that we have proved the following fact: Let the nodes $\mathbf{x}$ and $\mathbf{y}$ be optimal of the type $(\nu_1,..., \nu_n)$ and $(\nu_1,..., \nu_{k-1}, \mu_k, \nu_{k+2},..., \nu_n)$, respectively, where $\mu_k = \min(r, \nu_k + \nu_{k+1})$. Suppose that $\mu_k > \nu_k$. Then $R(\mathbf{x}) < R(\mathbf{y})$. This observation implies

COROLLARY 1. *The optimal quadrature formula of the type*

$$\underbrace{(1,..., 1)}_{N}$$

in the class $W_\infty^r[a, b]$ *has a minimal error amont all optimal quadrature formulas of the type* $(\nu_1,..., \nu_n)$ *in* $W_\infty^r[a, b]$, *where* $[(\nu_1 + 1)/2] + \cdots + [(\nu_h + 1)/2] \leq N$.

The main result of this paper was meanwhile extended by the author [24] to the classes $W_p^r[a, b]$ $(1 < p < \infty)$.

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